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Existence and uniqueness of positive periodic solutions of functional differential equations [☆]

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Abstract

In this paper we consider the existence and uniqueness of positive periodic solution for the periodic equation $y'(t) = -a(t)y(t) + \lambda h(t)f(y(t - \tau(t)))$. By the eigenvalue problems of completely continuous operators and theory of α -concave or $-\alpha$ -convex operators and its eigenvalue, we establish some criteria for existence and uniqueness of positive periodic solution of above functional differential equations with parameter. In particular, the unique solution $y_\lambda(t)$ of the above equation depends continuously on the parameter λ . Finally, as an application, we obtain sufficient condition for the existence of positive periodic solutions of the Nicholson blowflies model.

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1. Introduction

Functional differential equations with periodic delays appear in some ecological models. For example, the model of the survival of red blood cells in an animal [19], and the model of “dynamic disease” [13], and so on. One of the important questions is whether these equations can support positive periodic solutions. In recent years, periodic popula-

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tion dynamics has become a very popular subject, and several different periodic models have been studied by many authors; see [1–3, 7–12, 16–18] and references therein.

In this article, we will study the existence and uniqueness of positive periodic solution of the first order functional differential equation with a parameter

$$y'(t) = -a(t)y(t) + \lambda h(t)f(y(t - \tau(t))), \quad (1.1)$$

where $a = a(t)$ and $\tau = \tau(t)$ are continuous T -periodic functions and $h = h(t)$ is a positive continuous T -periodic function. We also assume that $T, \lambda > 0$, and that $a = a(t)$ with $\int_0^T a(u) du > 0$. In particular, our equation can be interpreted as a standard Malthus population model $y' = -a(t)y$ subject to a perturbation with periodical delay.

Recently, Cheng and Zhang [1], Jiang et al. [12] (n -dimensional case), Wang and Kuang [15] investigated the existence of positive solutions of Eq. (1.1) by using Krasnosel'skii fixed point theorem. They showed that there exists one or two positive periodic solutions for (1.1) if $f_0 = 0$ and $f_\infty = \infty$, or $f_0 = \infty$ and $f_\infty = 0$, or $f_0 = l$ ($0 < l < \infty$) and $f_\infty = L$ ($0 < L < \infty$), where

$$f_0 = \lim_{u \rightarrow 0^+} \frac{f(u)}{u} \quad \text{and} \quad f_\infty = \lim_{u \rightarrow \infty} \frac{f(u)}{u}.$$

That is to say, all criteria obtained are depend on both f_0 and f_∞ . But for the case that criteria depend on one of f_0 and f_∞ , it remains open. In [20], by using supersolution and subsolution method, Zhang and Cheng studied the existence and nonexistence of positive solutions of Eq. (1.1) under the conditions: f is continuous and nondecreasing on $[0, +\infty)$, $f(0) > 0$ and $f_\infty = +\infty$, they obtained some existence results. However, their results are not valid for the Nicholson blowflies model

$$N'(t) = -\delta N(t) + PN(t - \tau)e^{-aN(t - \tau)}, \quad (1.2)$$

because $f(0) = 0$ and f does not satisfy the above monotonic condition. In the model (1.2), $\delta, P, a, \tau \in (0, \infty)$, $N(t)$ is the size of the population at time t , P is the maximum per capita daily egg production, $1/a$ is the size at which the population reproduces at its maximum rate, δ is the per capita daily adult death rate, and τ is the generation time [4, 14].

On the other hand, many authors investigated the existence and nonexistence of positive periodic solutions for Eq. (1.1), but, to the best of our knowledge, there is no result for the uniqueness of positive T -periodic solution of Eq. (1.1).

Motivated by the above questions, in this paper, we will establish several sufficient conditions for the existence of positive T -periodic solutions of Eq. (1.1) by using the eigenvalue theory of operators. Further, we also obtain sufficient conditions for the uniqueness of positive T -periodic solutions of Eq. (1.1) by using the theory of α -concave operator. In particular, the unique solution $y_\lambda(t)$ of (1.1) depends continuously on the parameter, i.e., $y_\lambda(t)$ is continuous and strong increasing in λ . That is, $\lambda_1 > \lambda_2 > 0$ implies $y_{\lambda_1}(t) \gg y_{\lambda_2}(t)$ and $\lim_{\lambda \rightarrow \lambda_0} \|y_\lambda(t) - y_{\lambda_0}(t)\| \rightarrow 0$ for $t \in [0, T]$. In addition $\lim_{\lambda \rightarrow 0^+} \|y_\lambda(t)\| = 0$, $\lim_{\lambda \rightarrow +\infty} \|y_\lambda(t)\| = +\infty$ for any $t \in [0, T]$. Finally, as an application to the Nicholson blowflies model (1.2), we obtain sufficient condition for the existence of positive periodic solutions.

2. Some lemmas

In this section we shall list some lemmas, which are important to prove our results.

By (1.1), we have

$$y(t) = \lambda \int_t^{t+T} G(t, s) h(s) f(y(s - \tau(s))) ds, \quad (2.1)$$

where

$$G(t, s) = \frac{\exp(\int_t^s a(u) du)}{\exp(\int_0^T a(u) du) - 1}, \quad s \in [t, t+T].$$

It is easy to see that Eq. (1.1) has T -periodic solutions if and only if (2.1) has T -periodic solutions [1]. Further, we have

$$0 < N \equiv \min_{t \in [0, T]} G(t, s) \leq G(t, s) \leq \max_{t \in [0, T]} G(t, s) \equiv M, \quad t \leq s \leq t+T,$$

and

$$1 \geq \frac{G(t, s)}{\max_{t \in [0, T], s \in [t, t+T]} G(t, s)} \geq \frac{\min_{t \in [0, T], s \in [t, t+T]} G(t, s)}{\max_{t \in [0, T], s \in [t, t+T]} G(t, s)} = \frac{N}{M} = k > 0.$$

Let X be the set of all real T -periodic continuous functions, endowed with the usual linear structure as well as the norm

$$\|y\| = \sup_{t \in [0, T]} |y(t)|.$$

Then X is a Banach space. Define two cones of X by

$$P_1 = \{y(t) \in X: y(t) \geq k\|y\|, t \in \mathbb{R}\}$$

and

$$P_2 = \{y(t) \in X: y(t) \geq 0, t \in \mathbb{R}\},$$

and an operator $A: X \rightarrow X$ by

$$(Ay)(t) = \int_t^{t+T} G(t, s) h(s) f(y(s - \tau(s))) ds. \quad (2.2)$$

Then

$$(Ay)(t) \leq M \int_t^{t+T} h(s) f(y(s - \tau(s))) ds \quad (2.3)$$

and

$$(Ay)(t) \geq N \int_t^{t+T} h(s) f(y(s - \tau(s))) ds \geq k\|Ay\|. \quad (2.4)$$

Lemma 1 [1]. $AP_1 \subseteq P_1$.

In order to prove our results, the following results and some definitions are needed. Let E be a real Banach space, P is a cone of E . The semi-order induced by the cone P denoted by “ \leq .” That is, $x \leq y$ if and only if $y - x \in P$.

Definition 1 [6]. P is a cone of a real Banach space E . P is a solid cone, if P° is not empty, where P° is the interior of P .

Definition 2 [6]. P is a solid cone of a real Banach space E and $A : P^\circ \rightarrow P^\circ$ is an operator. A is called an α -concave operator ($-\alpha$ -convex operator), if

$$A(tx) \geq t^\alpha Ax \quad (A(tx) \leq t^{-\alpha} Ax) \quad \text{for any } x \in P^\circ \text{ and } 0 < t < 1,$$

where $0 \leq \alpha < 1$. The operator A is increasing (decreasing), if $x_1, x_2 \in P^\circ$ and $x_1 \leq x_2$ imply $Ax_1 \leq Ax_2$ ($Ax_1 \geq Ax_2$), and further, the operator A is strong increasing (decreasing), if $x_1, x_2 \in P^\circ$ and $x_1 < x_2$ imply $Ax_2 - Ax_1 \in P^\circ$ ($Ax_1 - Ax_2 \in P^\circ$). Let x_λ be a proper element of the enginvalve λ of A , that is $Ax_\lambda = \lambda x_\lambda$. x_λ is called strong increasing (decreasing), if $\lambda_1 > \lambda_2$ implies $x_{\lambda_1} - x_{\lambda_2} \in P^\circ$ ($x_{\lambda_2} - x_{\lambda_1} \in P^\circ$), which is denoted by $x_{\lambda_1} \gg x_{\lambda_2}$ ($x_{\lambda_2} \gg x_{\lambda_1}$).

Lemma 2 [5,21]. Suppose D is an open subset of the an infinite-dimensional real Banach space E , $\theta \in D$, and P is a cone of E . If the operator $\Gamma : P \cap \bar{D} \rightarrow P$ is completely continuous with $\Gamma\theta = \theta$ and satisfies

$$\inf_{x \in P \cap \partial D} \|\Gamma x\| > 0,$$

then Γ has a proper element on $P \cap \partial D$ associated with a positive eigenvalue. That is, there exist $x_0 \in P \cap \partial D$ and $\mu_0 > 0$ such that $\Gamma x_0 = \mu_0 x_0$.

Lemma 3 [6]. Suppose P is a normal solid cone of a real Banach space, $A : P^\circ \rightarrow P^\circ$ is an α -concave increasing (or α -convex decreasing) operator. Then A has only one fixed point in P° .

3. Existence of positive solutions

In this section we shall establish some sufficient conditions for the existence of positive solutions of Eq. (1.1). Before stating our results, we need to impose conditions on the function f . Assume that

$$f : [0, +\infty) \rightarrow [0, +\infty) \quad \text{is continuous with } f(0) = 0. \quad (3.1)$$

We call a continuously differentiable and T -periodic function a periodic solution of (1.1) associated with λ^* if it satisfies (1.1) when $\lambda = \lambda^*$. Set

$$\int_0^T h(s) ds = B$$

and

$$M(r) = \max_{0 \leq u \leq r} f(u) \quad \text{and} \quad m(r) = \min_{kr \leq u \leq r} f(u).$$

Then $B > 0$.

Theorem 1. Suppose that (3.1) holds and that $0 < f_\infty < +\infty$. Then there exists a positive number R_0 such that for any $r > R_0$, (1.1) has a positive T -periodic solution $y_r^*(t)$ associated with $\lambda^* \in [\lambda_1, \lambda_2]$ and $\|y_r^*(t)\| = r$, where λ_1 and λ_2 are two positive finite numbers.

Proof. By (2.1) and (2.2), (1.1) has a positive T -periodic solution $y_r^*(t)$ associated with $\lambda^* > 0$ if and only if the operator A has a proper element y_r^* associated with the eigenvalue $1/\lambda^* > 0$. Since $0 < f_\infty < +\infty$, there exist $l_2 > l_1 > 0$ and large enough $\eta > 0$ such that

$$l_1 u < f(u) < l_2 u \quad (3.2)$$

for any $u \geq \eta$. Let $R_0 = k^{-1}\eta$ and

$$\Omega_r = \{y \in X : \|y\| < r\},$$

where $r > R_0$. Then Ω_r is a bounded open subset of the Banach space X and $\theta \in \Omega_r$. Together with Lemma 1, we note that $A : P_1 \cap \bar{\Omega}_r \rightarrow P_1$ is completely continuous with $A\theta = \theta$. In addition,

$$\begin{aligned} (Ay)(t) &= \int_t^{t+T} G(t, s)h(s)f(y(s - \tau(s)))ds \geq N \int_t^{t+T} h(s)f(y(s - \tau(s)))ds \\ &\geq l_1 N k \|y\| \int_0^T h(s)ds = l_1 N k B r > 0 \end{aligned}$$

for any $r > R_0$ and $y \in P_1 \cap \partial\Omega_r$. So we have

$$\inf_{y \in P_1 \cap \partial\Omega_r} \|Ay\| \geq l_1 N k B r > 0.$$

By Lemma 2, for any $r > R_0$, the operator A has a proper element $y_r^* \in P_1$ associated with the eigenvalue $\mu^* > 0$, further y_r^* satisfies $\|y_r^*\| = r$. Let $\lambda^* = 1/\mu^*$. Then (1.1) has a positive T -periodic solution $y_r^*(t)$ associated with λ^* .

From the proof above, for any $r > R_0$, there exists a positive T -periodic solution $y_r^* \in P_1 \cap \partial\Omega_r$ associated with $\lambda^* > 0$. That is,

$$y_r^*(t) = \lambda^* \int_t^{t+T} G(t, s)h(s)f(y_r^*(s - \tau(s)))ds \quad \text{with } \|y_r^*\| = r.$$

On the one hand,

$$y_r^*(t) \leq \lambda^* M \int_t^{t+T} h(s)f(y_r^*(s - \tau(s)))ds \leq \lambda^* M l_2 r \int_0^T h(s)ds \leq \lambda^* M l_2 B r$$

and further

$$\|y_r^*\| = r \leq \lambda^* l_2 M B r,$$

which means that

$$\lambda^* \geq \frac{1}{l_2 M B} = \lambda_1.$$

On the other hand,

$$\begin{aligned} y_r^*(t) &\geq \lambda^* N \int_t^{t+T} h(s) f(y_r^*(s - \tau(s))) ds \geq \lambda^* l_1 N k \|y_r^*\| \int_0^T h(s) ds \\ &= \lambda^* l_1 N B k r, \end{aligned} \quad (3.3)$$

then

$$\|y_r^*\| = r \geq \lambda^* l_1 N B k r,$$

which leads to

$$\lambda^* \leq \frac{1}{l_1 N B k} = \frac{1}{l_1 M B k^2} = \lambda_2.$$

In conclusion, $\lambda^* \in [\lambda_1, \lambda_2]$. The proof is complete. \square

Theorem 2. Suppose that (3.1) holds and that $f_\infty = +\infty$. Then there exists a positive number R_0 such that for any $r > R_0$, (1.1) has a positive T -periodic solution $\check{y}^*(t)$ satisfying $\|\check{y}_r^*\| = r$ associated with $\check{\lambda}^* \leq \check{\lambda}$, where $\check{\lambda}$ is a positive finite number.

Similarly to the proof of Theorem 1, it is easy to see from (3.2) and (3.3) that Theorem 2 is also true.

Theorem 3. Suppose that (3.1) holds and that $0 < f_0 < +\infty$. Then there exists a positive number r_0 such that for any $0 < r < r_0$, (1.1) has a positive T -periodic solution $\hat{y}_r(t)$ satisfying $\|\hat{y}_r\| = r$ associated with $\hat{\lambda} \in [\hat{\lambda}_1, \hat{\lambda}_2]$, where $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are two positive finite numbers.

Proof. Since $0 < f_0 < +\infty$, there exist $r_0 > 0$ and constants $c_2 > c_1 > 0$ such that

$$c_1 u < f(u) < c_2 u$$

for any $0 < u < r_0$. Set

$$U_r = \{y \in X: \|y\| < r\},$$

where $0 < r \leq r_0$. Then U_r is a bounded open subset of the Banach space X and $\theta \in U_r$. Together with Lemma 1, note that $A: P_1 \cap \bar{U}_r \rightarrow P_1$ is completely continuous with $A\theta = \theta$ and

$$\begin{aligned}
Ay(t) &= \int_t^{t+T} G(t, s)h(s)f(y(s - \tau(s))) ds \geq N \int_t^{t+T} h(s)f(y(s - \tau(s))) ds \\
&\geq c_1 Nk \|y\| \int_0^T h(s) ds = c_1 Nk B r > 0
\end{aligned}$$

for any $0 < r < r_0$ and $y \in P_1 \cap \partial U_r$. Thus

$$\inf_{y \in P_1 \cap \partial U_r} \|Ay\| \geq c_1 Nk B r > 0.$$

By Lemma 2, for any $0 < r < r_0$, the operator A has a proper element $\hat{y}_r \in P_1$ associated with the eigenvalue $\hat{\mu} > 0$, further \hat{y}_r satisfies $\|\hat{y}_r\| = r$. Let $\hat{\lambda} = 1/\hat{\mu}$ and follow the proof of Theorem 1; we complete the proof of Theorem 3. \square

Theorem 4. Suppose that (3.1) holds and that $0 < f_0 = +\infty$. Then there exists a positive number r_0 such that for any $0 < r < r_0$, (1.1) has a positive T -periodic solution $\hat{y}_r(t)$ satisfying $\|\hat{y}_r\| = r$ associated with $\hat{\lambda} \leq \hat{\lambda}_*$, where $\hat{\lambda}_*$ is a positive finite number.

The proof is similar to that of Theorem 3, we omit it here.

Theorem 5. Suppose that (3.1) holds and that there exist $\bar{r} > 0$ and $c_{\bar{r}} > 0$ such that $m(\bar{r}) \geq c_{\bar{r}} > 0$. Then (1.1) has a positive T -periodic solution $y_{\bar{r}}(t)$ satisfying $\|y_{\bar{r}}\| = \bar{r}$ associated with $\bar{\lambda} \geq \bar{\lambda}_*$, where $\bar{\lambda}_*$ is a positive finite number.

Proof. Set

$$V_{\bar{r}} = \{y \in X : \|y\| < \bar{r}\}.$$

Note that $V_{\bar{r}}$ is a bounded open subset of the Banach space X and $\theta \in V_{\bar{r}}$, and $A : P_1 \cap \bar{V}_{\bar{r}} \rightarrow P_1$ is completely continuous with $A\theta = \theta$. In addition,

$$\inf_{y \in P_1 \cap \partial V_{\bar{r}}} \|Ay\| \geq c_{\bar{r}} Nk B > 0.$$

By Lemma 2, our result required. The proof is complete. \square

We now consider the existence of positive T -periodic solution for the well-known model (1.2), which is the special case of (1.1). Assume $N(t)$ is the periodic solution of (1.2), we have

$$N(t) = \int_t^{t+T} H(t, s) P f(N(s - \tau)) ds,$$

where $f(u) = ue^{-au}$,

$$H(t, s) = \frac{e^{\delta(s-t)}}{e^{\delta T} - 1}, \quad s \in [t, t + T],$$

and

$$\frac{\min_{t \in [0, T], s \in [t, t+T]} H(t, s)}{\max_{t \in [0, T], s \in [t, t+T]} H(t, s)} = \frac{N'}{M'} = \frac{1/(e^{\delta T} - 1)}{e^{\delta T}/(e^{\delta T} - 1)} = \frac{1}{e^{\delta T}} = k.$$

4. Uniqueness of positive solution

In the previous section, we have obtained some existence criteria for positive periodic solutions of Eq. (1.1). Now we further consider the uniqueness of positive periodic solution of Eq. (1.1).

Theorem 6. Suppose that $f(u) : [0, +\infty) \rightarrow [0, +\infty)$ is a nondecreasing function with $f(u) > 0$ for $u > 0$, and satisfies $f(\rho u) \geq \rho^\alpha f(u)$, for any $0 < \rho < 1$, where $0 \leq \alpha < 1$. Then Eq. (1.1) has a unique positive T -periodic solution $y_\lambda(t)$. Furthermore, such a solution $y_\lambda(t)$ satisfies the following properties:

- (i) $y_\lambda(t)$ is strong increasing in λ . That is, $\lambda_1 > \lambda_2 > 0$ implies $y_{\lambda_1}(t) \gg y_{\lambda_2}(t)$ for $t \in [0, T]$.
- (ii) $\lim_{\lambda \rightarrow 0^+} \|y_\lambda(t)\| = 0$, $\lim_{\lambda \rightarrow +\infty} \|y_\lambda(t)\| = +\infty$ for any $t \in [0, T]$.
- (iii) $y_\lambda(t)$ is continuous with respect to λ . That is, $\lambda \rightarrow \lambda_0 > 0$, implies $\|y_\lambda(t) - y_{\lambda_0}(t)\| \rightarrow 0$ for any $t \in [0, T]$.

Proof. Set $\Psi = \lambda A$, and A be the same as in (2.2). It is easy to see that P_2 is a normal solid cone of X , and its interior $P_2^\circ = \{y(t) \in X : y(t) > 0\}$. By (2.2)–(2.4), the operator Ψ maps P_2 into P_2 . In view of $h(s) > 0$, $f(u) > 0$ for $u > 0$, and $G(t, s) > 0$, it is easy to see that $\Psi : P_2^\circ \rightarrow P_2^\circ$. We assert that $\Psi : P_2^\circ \rightarrow P_2^\circ$ is an α -concave increasing operator. Indeed

$$\begin{aligned} \Psi(\rho y) &= \lambda \int_t^{t+T} G(t, s) h(s) f(\rho y(s - \tau(s))) ds \\ &\geq \lambda \rho^\alpha \int_t^{t+T} G(t, s) h(s) f(y(s - \tau(s))) ds \\ &= \lambda \rho^\alpha \Psi(y) \quad \text{for any } 0 < \rho < 1, \end{aligned}$$

where $0 \leq \alpha < 1$. Since $f(u)$ is nondecreasing, then

$$\begin{aligned} (\Psi y_*)(t) &= \lambda \int_t^{t+T} G(t, s) h(s) f(y_*(s - \tau(s))) ds \\ &\leq \lambda \int_t^{t+T} G(t, s) h(s) f(y_{**}(s - \tau(s))) ds \\ &= (\Psi y_{**})(t) \quad \text{for } y_* \leq y_{**}, \quad y_*, y_{**} \in X. \end{aligned}$$

In view of Lemma 3, Ψ has a unique fixed point $y_\lambda \in P_2^\circ$. The first part of our results is proved.

Next, we give the proof for (i)–(iii). Let $\mu = 1/\lambda$, and denote $\lambda Ay_\lambda = y_\lambda$ by $Ay_\mu = \mu y_\mu$. Assume $0 < \mu_1 < \mu_2$. Then $y_{\mu_1} \geq y_{\mu_2}$. Indeed, set

$$\bar{\eta} = \sup\{\eta: y_{\mu_1} \geq \eta y_{\mu_2}\}. \quad (4.1)$$

We assert $\bar{\eta} \geq 1$. If it is not true, then $0 < \bar{\eta} < 1$, and further

$$\mu_1 y_{\mu_1} = Ay_{\mu_1} \geq A(\bar{\eta} y_{\mu_2}) \geq \bar{\eta}^\alpha Ay_{\mu_2} = \bar{\eta}^\alpha \mu_2 y_{\mu_2},$$

which imply

$$y_{\mu_1} \geq \bar{\eta}^\alpha \frac{\mu_2}{\mu_1} y_{\mu_2} \gg \bar{\eta}^\alpha y_{\mu_2} \gg \bar{\eta} y_{\mu_2}.$$

This is a contradiction to (4.1).

In view of the discussion above, we have

$$y_{\mu_1} = \frac{1}{\mu_1} Ay_{\mu_1} \geq \frac{1}{\mu_1} Ay_{\mu_2} = \frac{\mu_2}{\mu_1} y_{\mu_2} \gg y_{\mu_2}. \quad (4.2)$$

Hence, $y_\mu(t)$ is strong decreasing in μ . Namely $y_\lambda(t)$ is strong increasing in λ . (i) is proved.

Set $\mu_2 = \mu$ and fix μ_1 in (4.2), we have $y_{\mu_1} \geq (\mu/\mu_1)y_\mu$, for $\mu > \mu_1$. Further

$$\|y_\mu\| \leq \frac{\mu_1 N_1}{\mu} \|y_{\mu_1}\|, \quad (4.3)$$

where $N_1 > 0$ is a normal constant. Notice that $\mu = 1/\lambda$, we have $\lim_{\lambda \rightarrow 0^+} \|y_\lambda(t)\| = 0$. Let $\mu_1 = \mu$, and fix μ_2 , again by (4.2) and normality of P_2 , we have $\lim_{\lambda \rightarrow +\infty} \|y_\lambda(t)\| = +\infty$. (ii) is required.

Next, we show the continuity of $y_\mu(t)$. For given $\mu_0 > 0$. By (i),

$$y_\mu \ll y_{\mu_0} \quad \text{for any } \mu > \mu_0. \quad (4.4)$$

Let $l_\mu = \sup\{v > 0 \mid y_\mu \geq v y_{\mu_0}, \mu > \mu_0\}$. Obviously, $0 < l_\mu < 1$, and $y_\mu \geq l_\mu y_{\mu_0}$. So, we have

$$\mu y_\mu = Ay_\mu \geq A(l_\mu y_{\mu_0}) \geq l_\mu^\alpha Ay_{\mu_0} = l_\mu^\alpha \mu_0 y_{\mu_0},$$

and further

$$y_\mu \geq \frac{\mu_0}{\mu} l_\mu^\alpha y_{\mu_0}.$$

By the definition of l_μ ,

$$\frac{\mu_0}{\mu} l_\mu^\alpha \leq l_\mu \quad \text{or} \quad l_\mu \geq \left(\frac{\mu_0}{\mu}\right)^{1/(1-\alpha)}.$$

Again by the definition of l_μ , we have

$$y_\mu \geq \left(\frac{\mu_0}{\mu}\right)^{1/(1-\alpha)} y_{\mu_0} \quad \text{for any } \mu > \mu_0. \quad (4.5)$$

Notice that P_2 is a normal cone, in view of (4.4) and (4.5), we obtain

$$\|y_{\mu_0} - y_{\mu}\| \leq N_2 \left[1 - \left(\frac{\mu_0}{\mu} \right)^{1/(1-\alpha)} \right] \|y_{\mu_0}\| \rightarrow 0, \quad \mu \rightarrow \mu_0 + 0.$$

In the same way

$$\|y_{\mu} - y_{\mu_0}\| \rightarrow 0, \quad \mu \rightarrow \mu_0 - 0,$$

where $N_2 > 0$ is a normal constant. Consequently, (iii) holds. \square

Remark 1. In Theorem 5, the operator does not need to be completely continuous even continuous, and the function f satisfying the conditions of Theorem 5 can be easily found. For example,

$$f(u) = u^{\alpha_1} + u^{\alpha_2} + \cdots + u^{\alpha_m},$$

where $\alpha_i > 0$ with $\sup_i \alpha_i < 1$, m is a positive integer.

Remark 2. Suppose that $f(u) : [0, +\infty) \rightarrow [0, +\infty)$ is a nonincreasing function with $f(u) > 0$ for $u > 0$, and satisfies $f(\rho u) \leq \rho^{-\alpha} f(u)$ for any $0 < \rho < 1$, where $0 \leq \alpha < 1$. In view of Lemma 3 again, we also obtain that the Eq. (1.1) has a unique positive T -periodic solution $y_{\lambda}(t)$. Further, (i)–(iii) of Theorem 5 hold. We also can take the function

$$f(u) = (u^{\alpha_1} + u^{\alpha_2} + \cdots + u^{\alpha_m})^{-1},$$

as an example, where $\alpha_i > 0$ with $\sup_i \alpha_i < 1$, m is a positive integer.

5. Application to the Nicholson blowflies model

In this section, we apply our results in Section 3 to the Nicholson blowflies model (1.2) and establish a existence criterion for positive solutions.

Theorem 7. Suppose $1 - e^{-\delta T} < PT \leq e(1 - e^{-\delta T})$. Then (1.2) has a positive T -periodic solution $N(t)$ such that

$$\|N\| = \frac{1}{a} \ln \left(\frac{PTe^{\delta T}}{e^{\delta T} - 1} \right).$$

Proof. Set

$$(\Gamma N)(t) = \int_t^{t+T} H(t, s) Pf(N(s - \tau)) ds.$$

Let X and P_1 be with the same meanings as in Section 2. If the operator Γ has a positive proper element $N \in X$ associated with the eigenvalue $\bar{\mu} = 1$, that is,

$$\Gamma N = \bar{\mu} N, \tag{5.1}$$

then (1.2) has a positive T -periodic solution $N(t)$. Notice that $f(u) = ue^{-au}$ is increasing on $[0, 1/a]$, satisfies (3.1) and $f(u) > 0$ for $u \in (0, 1/a]$, in addition, $f(u)$ attains its maximum $f_M = (ae)^{-1}$ at $u = 1/a$. In view of the continuity of $f(u)$, there exists r_0 , $0 < \delta_0 < kr_0 < r_0 \leq 1/a$ such that $f(r_0) = r_0 e^{-ar_0} = C_{r_0} > \delta_0 e^{-a\delta_0} > 0$. Thus

$$\begin{aligned} m(r_0) &= \min_{kr_0 \leq u \leq r_0} f(u) = f(kr_0) = kr_0 e^{-akr_0} = kr_0^{1-k} r_0^k e^{-akr_0} \\ &= kr_0^{1-k} \{f(r_0)\}^k \geq kr_0^{1-k} C_{r_0}^k > 0. \end{aligned}$$

Let

$$W_r = \{N \in X: \|N\| < r_0\}.$$

$\Gamma: P_1 \cap \bar{W}_r \rightarrow P_1$ is completely continuous and satisfies all conditions of Theorem 5, which implies there exists $\bar{\mu} > 0$ such that (5.1) holds and $\|N\| = r_0$. Therefore,

$$\|\Gamma N\| = \bar{\mu} \|N\|, \quad (5.2)$$

or

$$\begin{aligned} \bar{\mu} r_0 &= \max_{t \in [0, T]} \int_t^{t+T} H(t, s) P f(N(s - \tau)) ds = PM' M(r_0) T = PM' f(r_0) T \\ &= PM' T r_0 e^{-ar_0}. \end{aligned} \quad (5.3)$$

In (5.3), if we take r_0 such that $r_0 = (1/a) \ln(PM'T)$ with $1 < PM'T \leq e$, then $\bar{\mu} = 1$ satisfies (5.2). Notice that $M' = e^{\delta T} / (e^{\delta T} - 1)$ and $0 < r_0 \leq 1/a$. The proof is complete. \square

Remark 3. If we take $PM'T = e$ or $PT = e(1 - e^{-\delta T})$, then (1.2) has a positive T -periodic solution $N(t)$ such that $\|N\| = 1/a$. Similarly, if $PM'T > e$, we also can choose a proper $r'_0 = (1/a) \ln(k^{-1} PT M') > r_0 = (1/a) \ln(PM'T)$ such that (1.2) has a positive T -periodic solution $N(t)$ and

$$\|N\| = r'_0 = \frac{1}{a} \ln \left(\frac{PT e^{\delta T}}{k e^{\delta T} - 1} \right).$$

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